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The Roads Not Taken: Graph Theory and Macroeconomic Regimes in Stock-flow Consistent Modeling*

by

Miguel Carrión Álvarez
Grupo Santander

Dirk Ehnts
Bard College Berlin

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* The manuscript has been greatly improved thanks to the careful reading of Frances Coppola, Miguel Puente, Daniel Detzer, and Gregor Semieniuk. Correspondence: Miguel.carrion.alvarez@gmail.com; d.ehnts@berlin.bard.edu.

Levy Economics Institute
P.O. Box 5000
Annandale-on-Hudson, NY 12504-5000
http://www.levyinstitute.org

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ABSTRACT

Standard presentations of stock-flow consistent modeling use specific Post Keynesian closures, even though a given stock-flow accounting structure supports various different economic dynamics. In this paper we separate the dynamic closure from the accounting constraints and cast the latter in the language of graph theory. The graph formulation provides (1) a representation of an economy as a collection of cash flows on a network and (2) a collection of algebraic techniques to identify independent versus dependent cash-flow variables and solve the accounting constraints. The separation into independent and dependent variables is not unique, and we argue that each such separation can be interpreted as an institutional structure or policy regime. Questions about macroeconomic regime change can thus be addressed within this framework.

We illustrate the graph tools through application of the simple stock-flow consistent model, or "SIM model," found in Godley and Lavoie (2007). In this model there are eight different possible dynamic closures of the same underlying accounting structure. We classify the possible closures and discuss three of them in detail: the "standard" Godley–Lavoie closure, where government spending is the key policy lever; an "austerity" regime, where government spending adjusts to taxes that depend on private sector decisions; and a "colonial" regime, which is driven by taxation.

Keywords: Stock-flow Consistent Models; Closures; Graph Theory; Macroeconomic Regimes; Methodology

JEL Classifications: E16, E17

1. INTRODUCTION

Stock-flow consistent (SFC) modeling is a framework for looking at the macroeconomy from a monetary or financial point of view.[1] The basic object of interest is the flow of funds between different sectors of the economy. It is generally accepted that Tobin's (1981) Nobel Lecture contains an enumeration of the basic elements of SFC modeling, namely precision regarding time, tracking of stocks, several assets and rates of return, modeling of financial and monetary policy operations, Walras's Law, and adding-up constraints. While many publications of SFC models have followed in the last 30 years, we find that there is a gap concerning the very basic structures.

SFC models have been developed by Post Keynesians, and some modeling decisions that have been taken are perceived to be essential features of SFC models. This regards the closure of the SFC model and, given what is autonomous, the behavioral equations. We do not argue that the literature is unaware of the possibility of different closures or chose the "wrong" closure. Many authors discuss their specific closures, often producing variations of the model with alternative closures, like fixed or flexible exchange rates.[2] In this paper we focus on different closures of the simple model of Godley and Lavoie (2007), which is more fundamental than the specifics of the exchange rate regime. We use graph theory and linear algebra techniques for model analysis and visualization. Then we proceed to discuss the difference between the stock-flow structure proper and the additional features necessary for it to be solved or simulated.

The idea of the economy as a graph, which is a representation of a set of objects to some extent linked to each other, is clearly present in the thinking of Axel Leijonhufvud (2012):

> Let me start by asking, what is your first association when somebody talks to you about "the economy"? Is the image you get: factories working, supermarkets full of people, busy Wall Street, or, what? For today's purpose, I would like you to think of the economic system, first, as a web of contracts, contracts and understandings among agents in the economy.

[1] For a survey of the literature, see Caverzasi and Godin (2013).
[2] See Godley and Lavoie (2006) or Duwicquet and Mazier (2010).

It is unclear to what extent Leijonhufvud is being metaphorical. In section three of this paper we make the correspondence between macroeconomic stock-flow modeling and graph concepts precise and operational. One contention of this paper that is borne out by graph analysis is that multiple possible closures of the same underlying SFC structure are possible, representing different institutional arrangements or policy regimes, and also different analytical frameworks. In particular, Godley and Lavoie (2007) give the distinct impression that the SFC framework itself is closely tied to their favored Post Keynesian closures, which is not the case. Cleanly separating the accounting constraints on macroeconomic models from the determination of behavior provides a neutral setting to approach theoretical debates on expectations, rational or otherwise, on microfoundations, as well as practical issues of policy and institutional design.

This paper is geared to an audience of economists, and we want to use some mathematical tools in our paper. Because the audience knows more economics than they do mathematics, economics should be used to illuminate the mathematics, and not conversely. Further, the use of similar mathematics in other disciplines such as physics adds nothing to the understanding by economists, only acting as evidence of the soundness of the abstract mathematics for applied purposes. Therefore the proper way to bring the graph metaphor into the paper is not as a prerequisite for doing economics, but as a novel way to represent what economists already know, with the added bonus (and that would be the argument for introducing mathematical concepts in the first place) that once economists are comfortable with the new representation they can take advantage of tools from the new domain, but always projecting (familiar) economic language onto (unfamiliar) mathematical language and not the other way around.[3] We also want to avoid falling afoul of what Eriksson (2012) calls the "nonsense math effect," so if what follows is over the reader's head that should not count in favor of the paper.

[3] A paper written for mathematicians would have a different flavor, namely "you already know everything about graphs, here's yet another thing you can model as a graph," and it would work as an introduction to macroeconomics where (unfamiliar) economic concepts are illuminated by (familiar) mathematical concepts. But that's not this paper.

2. THE SIMPLE MODEL

The simplest model in Godley and Lavoie (2007, ch.3), model SIM (for "simplest"), represents an economy without fixed capital accumulation or inventories of durable goods (which they call a "pure service economy"), and with government money as the sole financial asset. It is assumed that services are provided through firms, of which households are at once owners/employees and customers. Firms distribute their entire profit to households as wages/profits. The government purchases services by issuing money, which is a nonredeemable liability of the government, and collects taxes payable in money. Households may accumulate this circulating money as savings. Model SIM can be represented by the cash-flow specification in table 1.

Table 1. Cash-flow specification for model SIM

Cash flow	From	To	Stock	Price
consumption (c)	households	firms	N/A	N/A
wages (w)	firms	households	N/A	N/A
expenditures (e)	government	firms	N/A	N/A
taxes (t)	households	government	N/A	N/A
savings (s)	households	government	Cash (S)	1

We have a consumption cash flow *c* paid by households (H) to firms (F) in exchange for services. There is also a government (G) expenditure cash flow *g* paying for services from firms. The firms pay out their income as a wage-bill cash flow *w* to households (there is no distinction between firm owners and employees, or between wages and profits). The government collects a tax cash flow *t* from households. Households may accumulate savings in the form of a cash asset stock (S), which is a liability of the government.[4] The net saving cash flow *s* is of a different character from the rest of the rows of the cash-flow specification, as Godley and Lavoie (2007, 60) describe:

[4] Assuming there is no government default on the redeemability of cash, the only way the stock of savings can be reduced is if the households consume out of their savings or "dissave."

> The first […] lines of the transactions matrix describe the variables which correspond, in principle, to the components of the National Income and Product Accounts (NIPA) arranged as transactions between sectors and which take place in some defined unit of time, such as a quarter or a year. These are the transactions which are usually to be found in standard macroeconomic textbooks. The [last] line […] describes the changes in stocks of financial assets and liabilities which correspond, in principle, to the Flow-of-Funds Accounts and which are necessary to complete the system of accounts as a whole.

This means that net saving s, what Godley and Lavoie call the "change in the stock of money," can be represented as a cash flow from the household to the government sector. Cash has no price, or rather, the price of a unit of cash is *1* as it is the numeraire, implying $s = \Delta S$. Also, since the price of cash is constant by definition there are no capital gains on cash holdings. Another feature of cash that is not generic for financial assets is that it yields no interest.

To see the necessity of representing saving as a cash flow from households to government, consider the following cash balance equation for the household sector:

(net saving) = (wage bill) – (consumption) – (taxes net of transfers)

or, in symbols, $s = w - c - t$. Rearranging terms so that all coefficients are positive we obtain

$$w = c + t + s$$

where s is not an actual but a notional cash flow; saving is the difference between the inflows w and the outflows $c + t$, and it is accumulated as a cash asset holding $s = \Delta S$. Therefore,

$$w = c + t + \Delta S.$$

This discussion makes clear that saving is not necessarily an actual cash flow but an accounting entry to ensure, on the one hand, the equality of inflows and outflows at each sector and, on the other hand, that an increase in the asset stock S has a corresponding flow. Unlike capital gains (which in this case are absent because the price of cash is constant), cash flows corresponding to changes in asset stocks must be part of the cash-flow balance at each node. In fact, accumulation of non-cash assets usually involves an actual asset purchase. Cash is peculiar in that it circulates as money and there is no actual purchase involved in accumulating savings in the form of a cash stock.

As a result of this, the cash-flow specification matrix of table 1 has the peculiarity that there are two cash flows from households to the government sector, namely t and $s = \Delta S$. As was just discussed, cash, an asset of the household sector, is a (constant-price, undated) liability of the government. Thus the cash flow corresponding to an increase in the stock of cash must be from the household sector to the government sector, just like taxes (net of transfers). In fact, in modern monetary theory government money can be understood as a bearer tax credit.[5]

Apart from the fact that taxes don't result in the accumulation of any stock, there is another difference between the two cash flows from households to government, namely their behavioral/institutional determination. Taxes and transfers are determined institutionally, usually in the form of fractions of other tax flows or stocks in the economy, and thus act as passive "stabilizers" of the system. Tax policy decisions adopted in one time period would take effect in later periods though a change in the passive tax formula. By contrast, net saving is at least partly determined by the household sector's demand for savings, which is a behavioral relation quite distinct from tax rules. Therefore, there are sound reasons for having two cash flows from households to government even if it may seem like a redundant, therefore ugly, arrangement.

[5] Mosler (2012) writes: "Under a fiat monetary system, the government spends money and then borrows what it does not tax."

Table 2. Balance sheet of model SIM Godley and Lavoie (2007), table 3.1

Stock	Households	Firms	Government
Cash	S		-S

The information in the cash-flow specification table 1 can be rearranged as a balance-sheet matrix (table 2) and a cash-flow matrix (table 3).

Table 3. Transactions-flow matrix of model SIM Godley and Lavoie (2007), table 3.2

cash flow	households	firms	government
consumption (c)	-c	c	
wages (w)	w	-w	
expenditures (e)		g	-g
taxes (t)	-t		t
savings (s)	-ΔS		ΔS

As the price of the only asset in the model (cash) is constantly equal to *1*, there is no need to keep track of capital gains; the periodic change in the asset stock value is equal to the value of the net assets acquired in the period. Everything with a plus sign in the economy is balanced by something similar with a minus sign. Balance equations expressing the conservation of cash for each sector are obtained by summing the cash flows for each column and equating the sum to zero or, equivalently, by equating the sum of the cash flows into a sector to the sum of the cash flows out of the same sector:

$$w = c + t + \Delta S$$

$$c + g = w$$

$$t + \Delta S = g$$

We note that the wages *w* cash flow plays the role of GDP in this model. Because all the rows of the cash-flow table 3 add up to zero, the balance equations above also add up to zero, so one of them is always redundant. Godley and Lavoie (2007) spend a considerable amount of space counting variables and equations. Much of this can be avoided by judicious use of graph theory and linear algebra, as we shall now see.

3. SOME GRAPH CONCEPTS

The overall claim of the present document is that the Godley–Lavoie stock-flow consistent framework can be productively represented on the mathematical structure of a *graph*, to be defined presently. At its most general, an abstract *graph* is a collection of directed *edges* between *nodes*.[6] A *directed* edge has a *source* node and a *target* node. Both edges and nodes can carry labels, numerical or otherwise. Edges and nodes are *primitive notions* basically defined by the source and target relations.

Informally, an *economy* will be modeled as a graph, whose nodes are *sectors* into which the economy is decomposed. Nodes can carry *balance- sheet* data. In the limit, one can imagine a detailed model in which nodes are individual economic units (individuals, households, firms, institutions), as in the picture of the economy as a network of contracts conjured by the Leijonhufvud quotation in the first section. Economic *relations* between economic units or sectors are represented by directed *edges* between the corresponding nodes. Edges can be labelled by cash *flows*, or by financial *assets*. The informal correspondence is summed up in table 4.

[6] The concept we are describing is sometimes called a *quiver*, being a collection of *arrows*.

Table 4. Macroeconomics and graphs

SFC modeling	Graph theory
economy	graph
sectors	nodes
economic relations	edges
sector "from"	source node
sector "to"	target node
net worth	node labels
stocks	node labels
cash flows	edge labels
financial assets	edge labels

3.1 Formal Correspondence

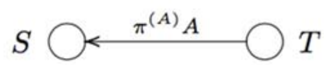
The form of the cash-flow specification matrix of a stock-flow consistent model (table A1) is reminiscent of the specification of a *graph*; in fact, it specifies two graphs: a cash-flow graph and a financial-asset graph, corresponding to the transactions-flow matrix (table 3) and the stock matrix (table 2), respectively. For reasons to be explained presently, the stock graph has the roles of source and target reversed with respect to the cash-flow graph (an asset purchase is a flow from the purchaser to the seller, while an asset itself is a liability of the seller to the purchaser).

It is hopefully clear that the columns "cash flow," "from," and "to" of a cash-flow specification matrix such as table A1 contain the data needed to specify the *source* and *target* relations of a *cash-flow graph* whose collection of nodes consists of those sectors appearing in the "from" or "to" columns of the table, and such that each row of the table represents a directed edge with source in the "from" node and target on the "to" node. Similarly, the columns "from," "to," and "stock" of table 1, restricted to the rows where the stock is nonempty, also specify a *graph* with the same nodes (sectors) but fewer edges (in this case, securities) as there isn't an edge of the stock graph associated with a row where the "stock" is absent.

3.1.1 Graphical representation

A graph (economy) can be represented diagrammatically by assigning to each node (sector) a dot or blob, and to each directed edge (economic relation) an arrow from its source node to its target node. Edge crossings are immaterial and may arise unavoidably as artifacts of the two-dimensional representation of sufficiently complex graphs.[7] Variables pertaining to sectors (nodes) or economic relations (edges) can be attached as labels to the corresponding blobs or arrows.

Figure 1. A stock as a relation between sectors



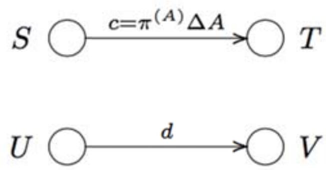
Stocks as edges: In the case of an SFC model, we can represent the rows of a cash-flow specification table (such as table 1) by arrows between blobs. As we know, the rows with nonempty stock variables form a subtable which we called "stock matrix" or "balance sheet" (table 2). Each security stock will be represented by an arrow from the blob representing the liability-issuing sector to the one representing the asset-holding sector, as in figure 1. Each arrow can be labelled by the monetary value (price times quantity) of the stock, which is more informative than just the name of the security the edge represents.

Flows as edges: The transactions-flow matrix (table 3) can also be represented as a graph with economic sectors as nodes and cash flows as edges. Because every security stock must have an associated net stock purchase cash flow, the cash-flow graph is at least as complex as the stock graph. However, because not every sector needs to hold security stocks and there are cash flows not representing an asset purchase, the cash-flow graph in general has both more nodes and more edges between the nodes than the stock graph. At this stage we don't represent stocks so we're only including the information from the transactions-flow matrix (table 3). Each edge of the

[7] Whether an abstractly specified graph can be represented graphically on a sheet of paper without edge crossings (in which case the graph is called *planar*) is an interesting topological problem but wholly irrelevant to the accounting.

graph will be labelled by a cash-flow variable, and each blob by the sector it represents, as in figure 2.

Figure 2. Cash flows between pairs of sectors

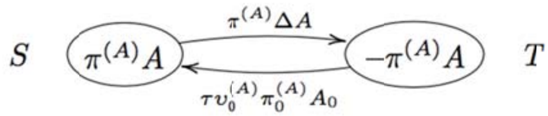


Here we have a cash flow c from sector S to sector T and a cash flow d from sector U to sector V. The cash flow $c = \pi^{(A)}\ \Delta A{:}S \rightarrow T$ represents a change in the stock of the security A, where $\pi^{(A)}$ is the unit price of A. The cash flow d is not associated with a stock change. Note also that the diagram "lives" at a generic time point, and in general we would need to attach a subscript t to all the stock, flow, and price variables to represent the time point at which the accounting is made.

Stocks on nodes: So far we have described a way to represent both the transactions-flow matrix and the stock matrix of an SFC model as separate graphs with the same set of nodes (sectors) but different edges (economic relations). Because the set of sectors is the same both stocks and flows could be represented in the same graph, but that would require keeping track of two different kinds of edges (stocks or flows) which would be confusing. An alternative to this is to represent security stocks by labels attached to the nodes of the cash-flow graph. For this purpose the most informative node label is the monetary value (price times quantity) of the security stock, with a negative sign for liabilities and a positive sign for assets.

A security will appear as a positive stock balance in the node representing the asset holder and a negative stock balance in a different node representing the liability issuer. Associated with any security there will be at least one net asset purchase cash flow from the asset holder to the liability issuer, but there can also be an interest payment cash flow going from the liability issuer to the asset holder. Interest payments are usually determined at the start of an accounting period and paid during it. The just-described system of an asset and a pair of cash flows would then be represented as in figure 3.

Figure 3. A stock and associated cash flows between two sectors



Here A is the stock quantity, $\pi^{(A)}$ the unit asset price at the end of the accounting period, and $\pi_0^{(A)}$ at the start, $\pi^{(A)}\,\Delta A{:}S \rightarrow T$ is the asset purchase cash flow, and $\tau v_o^{(A)}\,\pi_0^{(A)}\,A_0$ is the interest payment over a time period of length τ and with yield $v_o^{(A)}$ set at period start. The signs of the stock values $\pm\pi^{(A)}A$ on the nodes represent the fact that the security A is a liability of sector T (hence, a negative value) and an asset of sector S (a positive value).

3.2 Model SIM as Graph

As an example, the cash-flow specification table 3 defining model SIM can be interpreted as a pair of graph specifications, one for the asset stocks drawn as in figure 4 and the other for the cash flows drawn as in figure 5.

Figure 4. Stock graph for model SIM

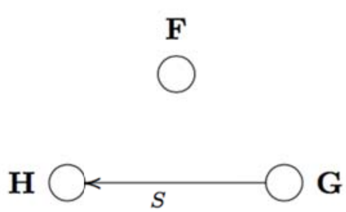
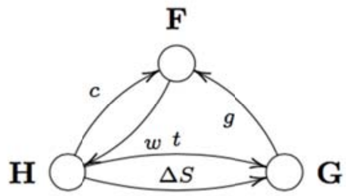
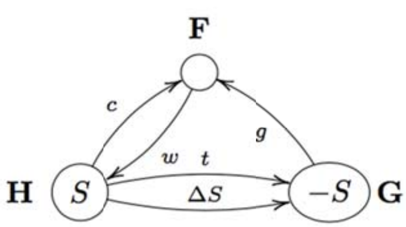


Figure 5. Cash-flow graph for model SIM



As explained above, the *S* stock arrow goes in the opposite direction of the *ΔS* arrow. This is because the stock *S* represents an asset of the household sector (H) and a liability of the government sector (G), so future redemptions (cash pays no interest) would make a negative contribution to the asset accumulation of *ΔS*. The firms sector is labelled (F). In any case, it can be confusing to represent both cash flows and financial stocks as arrows on the same collection of sectors (nodes). We prefer instead to put stock labels in the sectors (nodes), which then allows us to represent stocks and flows in the same diagram as in figure 6.

Figure 6. Stock and flow graph for model SIM



3.3 Spanning Trees and Dependent Flows

When working with a transactions-flow matrix such as that for model SIM (table 5) it becomes important to determine which cash flows can be deduced from which others by means of *conservation of cash flows* accounting rules. To this end, we now consider the economic interpretation of *spanning trees* and *elementary loops* of a cash-flow graph, illustrating the concepts with reference to model SIM.

Table 5. Transaction flow (table 3) with balance

cash flow	households	firms	government
consumption (c)	-c	c	
wages (w)	w	-w	
expenditures (e)		g	-g
taxes (t)	-t		t
savings (s)	-ΔS		ΔS
balance	w=c+t+ ΔS	c+g=w	t+ ΔS=g

A *spanning tree* is a subgraph including all the sectors (nodes) and one fewer economic relation (edges) than sectors (nodes), and is such that the subgraph is connected and has no loops. Adding any one edge to a spanning tree results in a closed *elementary loop*. In our setting, finding a spanning tree means selecting one fewer cash flow than the number of sectors, in such a way that every sector has at least one cash flow in or out of it, there are no closed loops, and adding an additional cash flow to the tree would close a loop. Each choice of spanning tree divides the cash flows into independent variables, which are the cash flows not on the spanning tree and the dependent variables (those on the spanning tree). The dependent variables are completely determined, via accounting relations at each node, by the independent ones.

3.3.1 A spanning tree example

In figure 7 an arbitrary choice of a spanning tree is drawn with solid arrows in the model SIM graph, and the economic edges not on the spanning tree are dotted.[8] We say the edges on the spanning tree are dependent because, by conservation of cash flows at each node, if all the cash flows corresponding to edges not in the tree are assumed to be zero then the flows on the tree must be zero. Put differently, the cash flows on a tree subgraph cannot be independent because a tree has no loops and thus flows on the tree "have nowhere to go."

[8] This is the choice made by Godley and Lavoie in their preferred closure of model SIM, but for the purposes of exposition of graph concepts the choice is by no means unique.

Associated elementary loops: The spanning tree of figure 7 is a connected graph with two edges linking the three nodes. Each of the edges not on the spanning tree, together with some edges of the spanning tree, defines a so-called elementary loop, as represented in figure 8. We say that edges not on the spanning tree represent independent variables because, as they form closed loops when combined with the tree, arbitrary cash flows on them do "have somewhere to go." In fact, the variables not on the tree fully determine the variables on the tree by conservation of cash flows at nodes.

Figure 7. A spanning tree in the model SIM graph

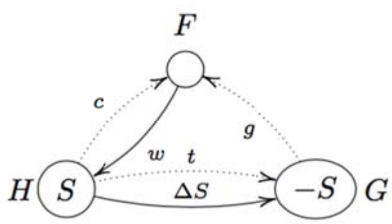
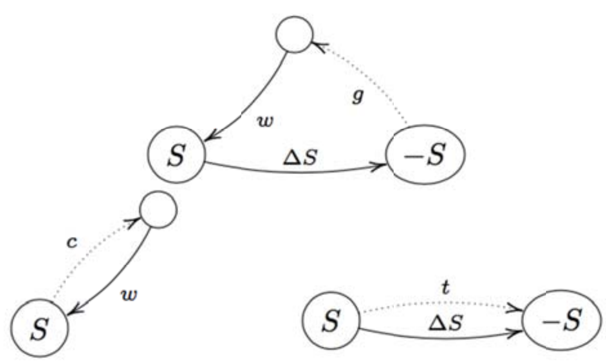


Figure 8. Elementary loops corresponding to figure 7



3.3.2 Beyond the "quadruple entry principle"

We observe that some of the elementary loops of figure 8 consist of only two economic relations (edges) linking two sectors (nodes), but that it is possible to have loops with a larger number of economic relations (edges) and sectors (nodes). What we are calling *elementary loops* are in fact a generalization of the *quadruple-entry principle* of Godley and Lavoie (2007, 47ff.), summarized in Caverzasi and Godin (2013, 5): “A concrete example of [Copeland’s] legacy is represented by the quadruple-entry system, which is a cardinal feature of today’s SFC models: that since someone’s inflow is someone else’s outflow, the standard double-entry system of accounting, in its social version, is doubled in a quadruple-entry system.”

Here we observe that an elementary loop involving only two cash flows does represent a “quadruple entry” contribution to the transactions-flow matrix, but elementary loops with three or more cash-flow edges are not necessarily reducible to combinations of two-edge loops and would be “sextuple entries” (or higher-order) of the transactions-flow matrix. In the case of the model SIM graph, elementary loops involving the government expenditure cash flow *g* necessarily consist of three cash flows. To illustrate the point, start with a collection of balanced cash flows as in table 3, and add δg to the government expenditure cash flow. The result is represented in table 6.

Table 6. A perturbation of model SIM

cash flow	households	firms	government
consumption (c)	-c	c	
wages (w)	w	-w	
expenditures (e)		g + **δg**	-g **-δg**
taxes (t)	-t		t
savings (s)	-ΔS		ΔS
balance	0	δg	-δg

This additional government expenditure δg breaks the accounting balance of the firms and government sectors. Because there is no other cash flow between these two sectors, the remaining sector of the economy (households) needs to be involved in order to restore accounting balance. This is why we said above any elementary loop involving the government expenditure cash flows must be more than quadruple entry. For instance, the excess income δg of the firms sector can be paid out as additional wages to the households sector, as in table 7.

Table 7. Propagation of additional government expenditure in model SIM

cash flow	households	firms	government
consumption (c)	-c	c	
wages (w)	W + **δg**	-w **-δg**	
expenditures (e)		g + **δg**	-g **-δg**
taxes (t)	-t		t
savings (s)	-ΔS		ΔS
balance	**δg**	0	**-δg**

The firms sector is back in accounting balance but the households and government sectors remain out of balance. However, unlike between government and firms, this time there are additional cash flows between households and government that can come into play to restore balance. For instance, households can accumulate the additional wages δg by saving, as in table 8. This closes a sextuple-entry accounting loop, corresponding to the three-edge loop at the top of figure 8. Figure 9 combines figures 7 and 8 to illustrate how the elementary loops given a tree generate the compatible flows on a network. This is summarized in table 9. Note that the elementary loops in figure 9 have been obtained from the particular choice of spanning tree in figure 7, but that the model SIM graph admits three more elementary loops represented in figure 10. These are associated to choices of spanning tree different from that of figure 7.

3.3.3 Linear algebra techniques

Using matrix algebra, we can write the balance equations associated with the transactions-flow matrix (table 3) in the following way:

Table 8. A sextuple-entry elementary loop in model SIM

cash flow	households	firms	government
consumption (c)	-c	c	
wages (w)	W + **δg**	-w **-δg**	
expenditures (e)		g + **δg**	-g **-δg**
taxes (t)	-t		t
savings (s)	-ΔS -**δg**		ΔS +**δg**
balance	0	0	0

Figure 9. Model SIM elementary loops as basis of compatible cash flow

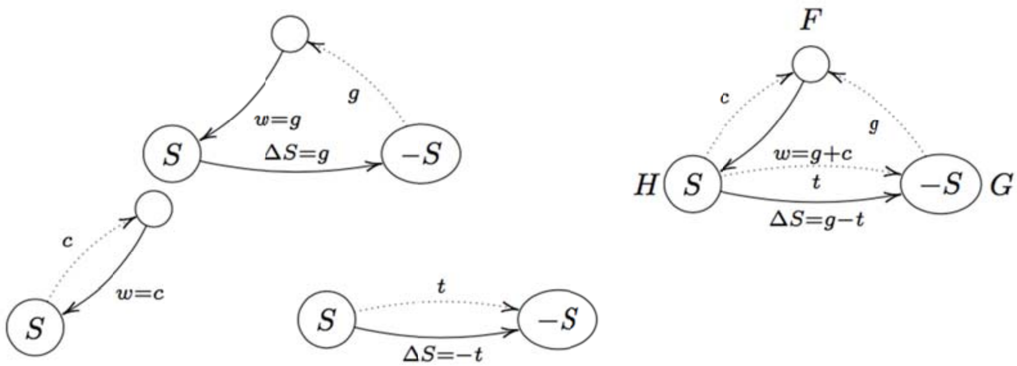
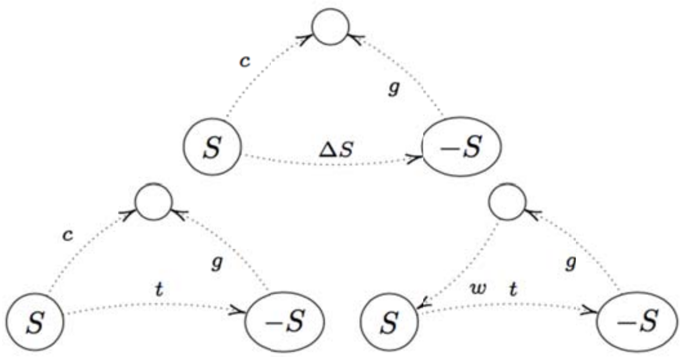


Table 9. Figure 9 as a transactions-flow matrix for model SIM

cash flow	households	firms	government
consumption (c)	-c	c	
wages (w)	g + c	-g -c	
expenditures (e)		g	-g
taxes (t)	-t		t
savings (s)	t - g		g - t

Figure 10. Elementary loops of model SIM not related to figure 7



$$\begin{pmatrix} c & w & g & t & \Delta S \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

The rectangular matrix has the same dimensions as the cash-flow matrix, with one row per cash flow and one column per sector, but the entries are simply ±*1* depending on whether the cash flow is incoming *(+)* or outgoing *(-)* at a sector. This is called the *incidence matrix* of the economy (graph), because a nonzero entry indicates an edge being incident on a node.

Because the rows of the incidence matrix add up to *0*, the number of independent columns (*rank*) of the matrix is one fewer than the number of columns.[9] In this case, the rank is *2* which means that in any set of three rows or columns, one of them is expressible as a linear combination of the other two. The rank is also the dimension of the largest invertible submatrix. Finding an invertible submatrix of maximal rank is equivalent to the selection of a spanning tree, as the rows of the invertible submatrix correspond to the economic relations (edges) of the spanning tree as discussed above.

For instance, our preferred choice of spanning tree corresponds to certain rows and columns of the matrices in the balance equation above, here highlighted:

[9] The mentioned properties of matrix rank are standard results from linear algebra.

$$\begin{pmatrix} c & \mathbf{w} & g & t & \mathbf{\Delta S} \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

Leaving the highlighted (spanning tree) variables on the left-hand side and shifting the unhighlighted (cycle-closing) variables to the right-hand side, changing the sign of the coefficients, results in:

$$\begin{pmatrix} w & \Delta S \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c & g & t \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

The left-hand side involves the incidence matrix of the spanning tree. Multiplying both sides on the right by the transpose of the incidence matrix $\left(\begin{smallmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{smallmatrix}\right)$ results in an invertible coefficient matrix on the left-hand side:

$$\begin{pmatrix} w & \Delta S \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} c & g & t \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ 1 & -2 \end{pmatrix}$$

Multiplying both sides again on the right by the inverse $\frac{1}{3}\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$,

$$\underbrace{\begin{pmatrix} w & \Delta S \end{pmatrix}}_{dependent} = \underbrace{\begin{pmatrix} c & g & t \end{pmatrix}}_{independent} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

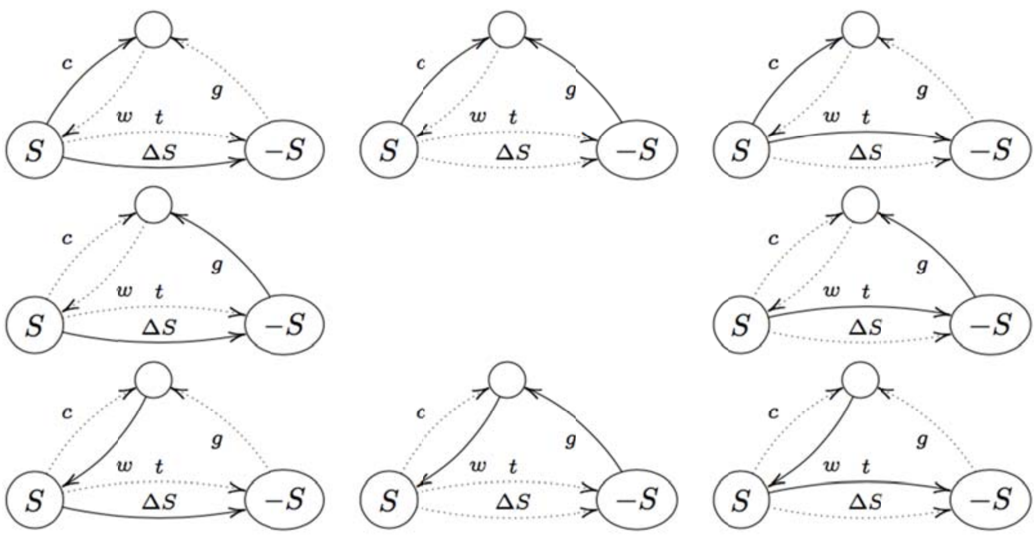
In other words, if we know the values of the variables (*c, g, t*), then the variables (*w*, ΔS) are completely determined by accounting, thus:

$w = c + g$

$\Delta S = g - t$

To *close* the system above we would have to specify behavioral or institutional equations for the independent variables. It would be overkill to use linear algebra to deduce the accounting relations. The point here is to illustrate the algorithmic nature of the process, which is readily generalized to large numbers of variables and equations. In the case of model SIM there are eight possible choices of two rows of the incidence matrix that are not proportional to each other. This means the economy (graph) admits exactly eight different spanning trees, as shown in figure 11, where the bottom-left tree is the choice of spanning tree we made above.

Figure 11. The eight spanning trees of model SIM



The enumeration of all the spanning trees of a graph is a combinatorial problem for which, again, a number of algorithms exist. For small graphs like that of model SIM it can be done by hand. The interest in enumerating the different spanning trees is not in counting how many there are but, as we will illustrate later, because it allows one to classify the various possible macroeconomic policy frameworks by those flows that are considered independent and those that are dependent.

4. SPANNING TREES AND POLICY REGIMES

One interpretation of the choice of a spanning tree is that of a macroeconomic policy regime. We will illustrate this with the example of model SIM from Godley and Lavoie (2007, ch.3). The eight possible choices of spanning tree can be classified by the two choices of independent flow variables in the pairs {c,w} associated with the "real economy," and {t, ΔS}, purely "monetary." Because two economic relations (edges) between the same two sectors (nodes) give rise to a loop, a spanning tree cannot contain both *c* and *w*, or both *t* and *ΔS*. Thus, either *c*, *w*, or *neither* is a dependent flow variables; similarly for *t* and *ΔS*. Conversely, either *c*, *w,* or *both* are independent flow variables, and similarly either *t*, *ΔS,* or both are independent flow variables. The number of independent variables must be three in order to close the model, and the government expenditure variable *g* will be either dependent or independent according to the character of the other four.

4.1 Real and Monetary Drivers

Some of the variables of the simple model are called independent because they are determined by behavioral relations. The other variables must then be called dependent because they are determined by accounting. The categories are mutually exclusive, and there is no third kind of variable. For instance, if taxation is determined as a percentage of income, taxation is an independent variable even though it changes "automatically" with income. It is called independent because it is not determined by accounting relations. If, on the other hand, taxes are adjusted to bring the government budget deficit to zero, then taxes become a dependent variable because they are determined by accounting relations. For instance, given the level of government spending and private consumption (which together determine income), there is only one level of taxes (net of transfers) that solves the model.[10]

The consumption/wage bill pair of cash flows represents the monetary realization of "real economy" flows, which are converted into monetary flows by means of factor and product prices. If *c* (consumption) is an independent flow variable, consumer demand is a driver of the economy

[10] In table 10 it is closure number 5, in the bottom-right corner, that describes this.

and we label the institutional structure of the economy as *demand-driven*. Similarly, if w (the wage bill) is an independent flow variable, employer demand for labor is a driver of the economy. We call this a *supply-driven* economy. It is possible for both consumption and the wage bill to be independent variables, and then we have an economy with both supply-driven and demand-driven characteristics.

Flows in the other dimension, that of taxation and saving, are purely monetary. It is possible that t (taxation) is an independent variable, which we term a "redistribution" institutional arrangement. The level of taxation then acts as a brake on consumption, thus bringing the economy to a halt. Alternatively, private demand for savings manifesting as realized saving ΔS could be a driver of the economy. A higher demand for savings translates into less consumption, again acting as a break on the economy. We call this a "rentier economy." Since savings can be used alternatively to run up wealth balances or run down debts, another interpretation could be named the "deleveraging economy." And, if both taxes and saving are independent variables, we call the setup a "financialized economy." Both the demand for savings from the rentiers and the taxes on households are restricting consumption and thus through their impact on wages (demand for labor) are able to impose unemployment on the economy.

4.2 Policy Regimes of Model SIM

Table 10 lists the choices of driving (independent) variables on the top and left sides of the table. The center cell would have four driving variables which we know is impossible for model SIM so this cell is not a possibility. In the case of the corner cells, there are only two driving variables listed on the top and left of the table, and so the government expenditure g flow is also a driving variable of the system. We have associated numbers to each of the eight choices of dependent and independent variables in model SIM. We now proceed to examine some of the choices more closely. Our choice has fallen on the closures with numbers 2, 4, and 7 for reasons that will become clear when we discuss the interpretation of these closures. In order to aid the intuition, we will call regime number 7 a "functional finance" economy, regime number 4 a "financialized" economy, and regime number 2 a "colonial" economy.

Table 10. Policy regimes of model SIM

	t	t, ΔS	*ΔS*
w	1	2	3
w, c	8	N/A	4
c	7	6	5

4.2.1 Regime 7: Functional finance economy

The policy regime that appears to be favored by Godley and Lavoie (2007) is the one on the bottom-left corner of figure 11 and table 10, where the driving variables are taxation t, consumption c, and, by implication, government expenditure g. Because taxation and government spending are both independent the government's fiscal position is fully independent and determines private saving ΔS. This does not mean that the households' desire for savings is determined by the government, but that actually realized saving is. In addition, the "real economy" is demand driven, with government expenditure and private consumption determining the wage bill w. We call this institutional arrangement the "functional finance economy" after Lerner (1943). The function of government spending is to change the real economy and should not be stopped by "traditional doctrine about what is sound or unsound" (Lerner 1943, 39). We have:

$$w = g + c$$
$$\Delta S = g - t$$

A closure of this system will require behavioral relations determining c and g, t, that is private consumption and the government's fiscal stance.

The standard Godley–Lavoie closure: The closure presented in Godley and Lavoie (2007, ch. 3) is as follows:

$$t = \theta w \qquad \theta<1$$
$$c = \alpha\,(w-t) + \beta S_0 \qquad 0<\beta<\alpha<1$$
$$g = (free)$$

where household income w is subject to a tax rate θ which can be negative, presumably to represent the possibility that transfers exceed taxes; consumption c depends on disposable income $w - t$ and the start-of-period stock of savings S_0 through coefficients α (household propensity to consume out of disposable income) and β (household propensity to consume out of savings); government expenditure g is a policy lever explicitly left unspecified.

Because model SIM is a pure service economy it is not possible for firms to accumulate inventory, and all demand $(c + g)$ is always met (there is no rationing), resulting in a wage bill $w = g + c$.

4.2.2 Regime 4: Financialized economy

If we demote government expenditure g to a dependent variable and replace taxation t with saving ΔS as a driver we obtain an economy in which the private sector is fully independent and the government fully subordinate: the private sector determines its consumption, wage bill w, and saving ΔS autonomously, and the government passively fills the consumption gap with its expenditure g and compensates any deviation of private saving from target by means of transfers which contribute to the net taxes t. As this is a situation in which the government has no fiscal stance of its own, the "financialized" label seems appropriate. The government's fiscal position is determined by the accounting relations:

$g = w - c$

$t = w - c - \Delta S$

The closure in this case is a specification of consumption, wages, and saving. These variables of private sector behavior fully determine the public sector stance. For instance, an increase in the target level of savings that the private sector wants to hold will reduce taxes. This opens up a road for models in the area of financialization, as in Hein (2009) and Stockhammer (2013). The redistribution of income via a less progressive tax code is now, via the target change in net saving, a major determinant of total taxes.

4.2.3 Regime 2: Colonial economy

Fadhel Kaboub has described a pedagogical experiment in which he requires his students to hand in a number of “Denison Volunteer Dollars” (DVD) for class credit, which students can obtain by volunteering at local charities. The economics department, posing as government, both issues the DVD to charities to pay for the volunteering and collects the DVD tax at the end of the course.[11] This arrangement is a lot like the pure-service economy of model SIM. There is a fixed tax t, which is an independent quantity decided independently by the government. The private sector can decide how much they want to work (possibly in excess of the tax liability) at a fixed wage rate, therefore the wage bill w is also an independent driver. Net saving ΔS are also a driver as the private sector can decide to save money for future taxes. In this way, consumption is a dependent variable, as is government expenditure (money will be issued on demand to pay for earned wages). Algebraically,

$c = w - \Delta S - t$

$g = \Delta S + t$

[11] Further information can be found at http://denison.edu/academics/economics/denison-volunteer-dollar-program.

The closure in this case is given by the tax, the associated wage bill, and the demand for savings. A less wholesome interpretation of this closure, motivating the “colonial economy” label, is the story told by Mosler (2014) about how the British got the Ghanaian tribesmen to work their plantations:

> They came up with this brilliant idea. They told everybody there was going to be a tax on their hut. It was called a hut tax. Everyone had to pay, what, 10 Pounds a month, something. Tax. Or they would get their house burned down by the British. What happened? Everybody said all right, what do we have to do to get the money to pay the tax? “Ah, if you come to the coffee plantation we’ll pay you one Pound a day to work.” Sure enough, people [started] coming over to work, to earn the money, so they didn’t get their house burned down. The tax, the monetary system, created the [employment].
>
> Then the British hired the people so they could get the money to pay the tax so they didn’t have their house burned down. They would spend the money first and pay people, and then collect the tax, right. And they spent more than they collected because some people saved them [Pounds] for paying taxes later.

Here the government again sets the tax, and freely provides work to meet the demand for wage income and cash savings.[12]

4.3 Explicit or Implicit Closures

One noteworthy feature of the standard Godley–Lavoie closure is that the independent variables t and c depend on w in the same period, which is in turn determined from them and g by the accounting relations $w=g+c$ and $\Delta S=g-t$. This circularity seems to contradict the classification of variables as dependent or independent. One could indeed envision a slight modification of the Godley–Lavoie closure of section 4.2.1 in which taxes in one period are a fraction of the income of the previous period, and similarly consumption is based on the previous period’s disposable income (in an “adaptive expectations” model), producing

$$t = \theta w_0$$

$$c = \alpha\,(w_0 - t_0) + \beta\, S_0$$

[12] See Forstater (2003) for a historical account of taxes in the context of colonization.

This would make each period's variables a linear function of the previous period's variables, plus the independent policy lever of government expenditure *g*:

$$t = \theta w_0$$
$$c = \alpha\, (w_0 - t_0) + \beta\, S_0$$
$$w = g + \alpha(w_0 - t_0) + \beta S_0$$
$$S = S_0 + g - \theta\, w_0$$

This is akin to the so-called explicit or forward Euler method in numerical analysis, where rates of change are calculated at the start of a period.[13] However, it is known that numerical stability is improved by basing rates of change on end-of-period variables, which leads to the so-called *implicit* or *backward Euler method*. The standard Godley–Lavoie closure, analogous to the backward method, need not be interpreted as economic agents being able to see the future (basing their consumption on unknown future variables) but as an artifact of time discretization.

4.4 The Meaning of *Ceteris Paribus*

One of the advantages of the spanning tree formulation is that it gives a precise meaning to the phrase *ceteris paribus*, and that meaning depends on the "analysis framework" (otherwise referred to as "policy regime"). For instance, as we know the Godley–Lavoie closure of section 4.2.1 is:

$$w = c + g$$
$$\Delta S = g - t$$

where the variables on the right-hand side are independent and determine the ones on the left-hand side. This means that the effect of a change in government expenditure *ceteris paribus*

[13] The forward Euler method for solving $f'(t) = F[f(t),t]$ is based on
$f(t) \approx f(t - \Delta t) + (\Delta t)f'(t - \Delta t)$ while the backward method is based on
$f(t) \approx f(t - \Delta t) + (\Delta t)f'(t)$.
The presence of $f'(t)$ rather than $f'(t-\Delta t)$ makes the method "implicit" as the unknown $f(t)$ to be solved for also appears on the right-hand side, whereas in the "explicit" method the right-hand side can be computed entirely with values at $t - \Delta t$.

causes a change in wages and saving. Why? Because *all other things being equal* must mean *all other independent things being equal* and, under the Godley–Lavoie closure, consumption and taxes are independent while wages and saving are not. Therefore, “changes in government expenditure, all other things being equal” means “changes in government expenditure at constant taxation and consumption.” Consider instead closure 3:

$$c = w - g$$
$$t = g + \Delta S$$

Under this closure wages (demand for labor) are independent and so is the demand for savings, while consumption and tax revenue are determined by wages and the demand for savings. In this case, a change in government expenditure *ceteris paribus* causes a change in consumption and tax revenue. This is entirely opposite to the effect under the Godley–Lavoie closure (what changed under the Godley–Lavoie closure are now independent variables and therefore constant *ceteris paribus*).

5. CONCLUSION

In this paper we introduced graph theory as a method for visualizing closures of SFC models. Hopefully, the new tool will enable SFC modelers to discuss their models more explicitly, especially given that sometimes the models can get very large. Graph theory allows modelers to visualize the model and facilitates discussion among researchers. We discussed different closures that are possible in the model SIM of Godley and Lavoie (2007). Although the model is simple indeed, already there are eight different ways to close it. We discussed the role of real and monetary drivers, and focused on three closures that we found particularly interesting. We hope to have provided a roadmap to explore the roads not taken in the SFC literature.

Another advantage is that explicit modeling of regime change is possible. How does an economy transition from socialism to capitalism? Perhaps by demoting government expenditure from an independent driver to a dependent stabilizer? How do government and money get "invented"? Start with behavioral equations saying that government spending equals taxation equals zero. The result is a system in which consumption is equal to wages and $w(t)$ [GDP] is the only time series that matters. Suppose an instability then develops on a date T. As an institutional innovation it is compensated by nonzero government spending and positive private saving, causing a stock of savings to appear. From that point on it is decided that wages and the stock of savings will be taxed.

The economy, as Leijonhufvud (2012) said, is a web of contracts that is constantly reshaped, sometimes by historical mistakes, sometimes by building institutions, sometimes by changes in individual behavior. It thus seems fair to say that there is an advantage from exploring many different roads instead of focusing on only one and forgetting the others until they are overgrown.

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